Stochastic parametric amplification due to higher order correlations: A perturbative approach to non-Abelian effects in time ordering

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In this paper, we use quantum-mechanical formalism to describe the time evolution of a classical dynamical system with fluctuating parameters. By appropriate choice of "interaction picture representation," and the use of the Baker-Campbell-Hausdorff formula in the chronological time ordered evolution, we have obtained analytical expressions for the Lyapunov exponent of the energy evolution of the dynamical system. Our approach proved to be very powerful in handling either stochastic or highly correlated processes. The approach lends itself to generalizations for use in a wide field of applications.

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I. INTRODUCTION

Lately, there is a growing interest for the so-called stochastic resonances. Excitation effects of large amplitude are induced not from harmonically periodic forcing like in the usual resonance effects, but from small-amplitude noises. These effects appear in a wide range of physical systems [1–4], from the pure quantum level (nuclear magnetic resonance, quantum Hall effect, etc.) to macroscopic dynamical systems (e.g., atmospheric disturbances [5], catastrophe theory, etc.).

In the present paper, we analytically treat the time evolution of a simple classical dynamical system with fluctuating parameters. Thus, the Hamiltonian of the system is described by $H_0+h_s(t)$, where H_0 is the Hamiltonian of the unperturbed harmonic oscillator itself, while $h_s(t)$ is some generic perturbative part of the Hamiltonian, which incorporates the time-dependence of the parameters of the system. We attempt to find an approximate analytic description for the time evolution of the system. We shall actually be mainly concerned with the energy evolution of the system. Thus, we write the quantum-mechanical analogue of the equation for the evolution of the system, and then we separate what we shall call the "anti-Hermitian" part of the disturbance (to be defined rigorously in Sec. II), while incorporating the "Hermitian" part of the disturbance into H_0, which is by itself "Hermitian". Actually, aiming at the Lyapunov exponent for the energy evolution of the system, we shall make use of the Baker-Campbell-Hausdorff formula rather than the usual perturbation expansion.

The commutativity or not of the Hamiltonian with itself at different times is of major importance here. The chronological evolution will depend, of course, on the exact expansion with respect to the various orders of the correlation functions of $h_s(t)$. In the present paper, we shall be mainly concerned with the stochastic time-dependence of the parameters of the system.

Moreover, by interpreting the time variable as a coordinate variable, the results of the present paper could be useful in studying the band theory of amorphous materials, characterizing various types of disorder by the corresponding types of time noises.

The paper is organized as follows. In Sec. II we build the mathematical formulation that describes our approximation for the time evolution of a harmonic oscillator with time varying parameters. To illustrate how good the approximate scheme is, we use it in Sec. III to obtain an analytical expression for the Lyapunov exponent of the energy of an oscillator described by the Mathieu equation. The approximate formula is compared with the exact numerical evolution with excellent results. Then, in Sec. IV we perform the same kind of comparison but with various types of stochastic fluctuations describing the time varying parameters now. The approximate formula could be used to obtain analytical estimates of the ensemble average of the Lyapunov exponent for the energy. The corresponding comparisons with the exact ensemble averages are, again, very promising. Finally, in Sec. V we conclude by presenting a short review of applications of the approximate formula.

II. THE CLASSICAL OSCILLATOR IN THE INTERACTION PICTURE

We will study the simplest dynamical system, the one-dimensional classical harmonic oscillator. Even though we treat a problem of classical mechanics, we find it useful to use the quantum mechanical formalism. In this formalism the equation of motion for the unperturbed oscillator takes the following form:

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$$i\frac{d}{dt}|\Psi\rangle = H_0|\Psi\rangle = -\omega_0\begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix}|\Psi\rangle, \tag{1}$$

where the state vector $|\Psi\rangle$ corresponds to the phase-space vector

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \omega_0 x \\ \dot{x} \end{pmatrix}. \tag{2}$$

In this description the conservation of energy of the classical oscillator $E=\frac{1}{2}m(\omega_0^2x^2+\dot{x}^2)$ corresponds to the conservation of the quantum mechanical probability $\langle\Psi|\Psi\rangle$, which is assured by the Hermiticity of the quantum mechanical Hamiltonian $H_0=-\omega_0\sigma_2$, where σ_2 is the corresponding Pauli matrix. Throughout we put $m=1$ for simplicity.

Let us now assume that the frequency of the oscillator depends on time as $\omega(t)$. Then, the corresponding quantum mechanical Hamiltonian turns out to be

$$H = -\omega_0\begin{pmatrix} 0 & -\iota \\ \iota\dfrac{\omega^2}{\omega_0^2} & 0 \end{pmatrix}. \tag{3}$$

This Hamiltonian is now non-Hermitian and the non-Hermitian part of it leads to nonconservation of the energy. Since in this paper we shall be interested on the evolution of the energy of the classical system, we separate this time-depending Hamiltonian into a Hermitian H_{H} (still time-depending) and an anti-Hermitian part H_{A}. Thus,

$$H = H_{\mathrm{H}} + H_{\mathrm{A}} = -\omega_0 h(t)\sigma_2 - \omega_0 g(t)\iota\sigma_1, \tag{4}$$

where

$$h(t) = \frac{1}{2}\left(\frac{\omega^2(t)}{\omega_0^2} + 1\right),$$

$$g(t) = \frac{1}{2}\left(\frac{\omega^2(t)}{\omega_0^2} - 1\right). \tag{5}$$

The anti-Hermitian part of the Hamiltonian, H_{A}, the term proportional to $\iota\sigma_1$, is the one that leads to nonconservation of the energy of the classical mechanical system. Hence, the oscillator can either absorb or lose energy.

Now, in order to study the evolution of the energy of the system, we shall use a new interaction picture where the Hermitian part of the Hamiltonian, H_{H}, of the dynamical system, even though time-depending, will play the role of the unperturbed Hamiltonian while the anti-Hermitian part of the Hamiltonian, H_{A}, will play the role of the interaction Hamiltonian. Thus,

$$|\Psi^{(I)}\rangle = \mathcal{T}e^{\iota\int_0^t H_{\mathrm{H}}(t')\mathrm{d}t'}|\Psi\rangle. \tag{6}$$

In this picture, the interaction Hamiltonian is

$$H^{(I)} = (\mathcal{T}e^{-\iota\int_0^t H_{\mathrm{H}}(t')\mathrm{d}t'})^{\dagger}H_{\mathrm{A}}(\mathcal{T}e^{-\iota\int_0^t H_{\mathrm{H}}(t')\mathrm{d}t'}). \tag{7}$$

The time ordering product operator, denoted by $\mathcal{T}$ in the expressions above, could be omitted in our case since H_{H} is proportional to σ_2 only and, therefore, it commutes with itself at different times. Finally, the time evolution of $|\Psi^{(I)}\rangle$ is determined by the $U^{(I)}(0,T)$ propagator which is given by

$$U^{(I)}(0,T) = \mathcal{T}e^{-\iota\int_0^T H^{(I)}(t)\mathrm{d}t}. \tag{8}$$

In the expression above, $\mathcal{T}$, the time ordering product operator, is necessary since $H^{(I)}$ does not commute with itself at different times,

$$[H^{(I)}(t), H^{(I)}(t')] \neq 0. \tag{9}$$

Inserting the actual expressions for H_{H} and H_{A} that are presented in Eq. (4), in the formula for the interaction Hamiltonian, we get

$$\begin{aligned} H^{(I)} &= -\iota\omega_0 g(t)\begin{pmatrix} -\sin 2\phi(t) & \cos 2\phi(t) \\ \cos 2\phi(t) & \sin 2\phi(t) \end{pmatrix} \\ &= \iota\omega_0 g(t)(\sigma_3 \sin 2\phi(t) - \sigma_1 \cos 2\phi(t)), \end{aligned} \tag{10}$$

where

$$\phi(t) = \omega_0\int_0^t h(t')\mathrm{d}t' = \omega_0\left(t + \int_0^t \mathrm{d}t' g(t')\right). \tag{11}$$

Now, the whole interaction Hamiltonian is purely anti-Hermitian, as H_{A} itself, and it is a linear combination of only $\iota\sigma_1$ and $\iota\sigma_3$. In order to compute the propagator $U^{(I)}(0,T)$ one proceeds in the usual way by dividing the time interval $(0,T)$ into infinitesimal time intervals $\delta t_1, \delta t_2, \ldots, \delta t_N$, where $\delta t_i = t_i - t_{i-1}$, $t_0=0$, and $t_N=T$. Then, expression (8) takes the following form:

$$U^{(I)}(0,T) = \mathcal{T}\lim_{N\to\infty}\prod_{i=1}^{N} e^{-\iota H^{(I)}(t_i)\delta t_i}. \tag{12}$$

Based on the Baker [6]-Campbell [7]-Hausdorff [8] (BCH) formula (see also Refs. [9–11] for recent alternative applications),

$$\exp(A)\exp(B) = \exp\left(A + B + \frac{1}{2}[A,B] + O(3)\right), \tag{13}$$

where $O(3)$ denotes terms of order at least 3 with respect to A and B, the product of any such successive exponential terms is given by

$$\begin{aligned} &\exp[\omega_0 g_k(-c_k\sigma_1 + s_k\sigma_3)\delta]\exp[\omega_0 g_{k-1}(-c_{k-1}\sigma_1 + s_{k-1}\sigma_3)\delta] \\ &\quad = \exp[(-\omega_0(g_k c_k + g_{k-1}c_{k-1})\sigma_1 + \omega_0(g_k s_k + g_{k-1}s_{k-1})\sigma_3)\delta \\ &\quad\quad - \omega_0^2 g_k g_{k-1}(s_k c_{k-1} - c_k s_{k-1})\iota\sigma_2\delta^2 + O(\delta^3)], \end{aligned} \tag{14}$$

where g_i, c_i, s_i, δ are abbreviations for $g(t_i)$, $\cos 2\phi(t_i)$, $\sin 2\phi(t_i)$, δt_i, respectively. By gathering all the first- and second-order terms with respect to δ in computing the whole product of Eq. (8), we obtain the following approximate formula for the propagator (cf., Appendix A):

$$U^{(I)}(0,T) = \mathcal{T}e^{-i\int_0^T H^{(I)}dt} \simeq U_2^{(I)}(0,T)$$
$$= \exp\left(-\sigma_1 \int_0^T A_1(t')dt' + \sigma_3 \int_0^T A_3(t')dt' \right.$$
$$- \iota\sigma_2 \int_0^T \int_0^T [A_3(t')A_1(t'') - A_1(t')A_3(t'')]$$
$$\left. \times \Theta(t'-t'')dt'dt'' \right), \quad (15)$$

up to second-order correlations of the function $g(t)$, where

$$A_1(t) = \omega_0 g(t)\cos 2\phi(t),$$
$$A_3(t) = \omega_0 g(t)\sin 2\phi(t), \quad (16)$$

and $\Theta(x)$ is the step function. At this point it should be noted that even the first-order terms in the exponential include contributions of all higher order correlations through the dependence of $\phi(t)$ on $g(t)$; the great power of the above analytical formula is hidden right here. Alternatively, using basic formulae of the Pauli matrices, the exponential expression in Eq. (15) could be written in a more explicit form as

$$U_2^{(I)}(0,T)$$
$$= \begin{pmatrix} \cosh a + a_3\left(\frac{\sinh a}{a}\right) & -(a_1+a_2)\left(\frac{\sinh a}{a}\right) \\ -(a_1-a_2)\left(\frac{\sinh a}{a}\right) & \cosh a - a_3\left(\frac{\sinh a}{a}\right) \end{pmatrix}, \quad (17)$$

where

$$a_{1,3} = \int_0^T dt A_{1,3}(t),$$
$$a_2 = \int_0^T dt' \int_0^{t'} dt''[A_3(t')A_1(t'') - A_1(t')A_3(t'')],$$
$$a = \sqrt{a_1^2 + a_3^2 - a_2^2}. \quad (18)$$

The use of the complete BCH expansion formula is equivalent to the complete perturbation expansion sum, if everything converges nicely. However, when one uses partial sums, it may be advantageous to use one or the other expansion, depending on the special problem one is dealing with. For example, if $[[A,B],A]=[[A,B],B]=0$, the approximate formula $\exp(A)\exp(B)=\exp(A+B+(1/2)[A,B])$ becomes exact. This is almost the case we are dealing with in our problem. A and B are ϵ times quantities of order unity along with Pauli matrices and thus $[A,B]$ is of order ϵ^2, which is almost zero for small magnitudes of ϵ. Thus, the assumed smallness of $g(t)$ in the expansions above justifies the use of the BCH formula up to the lowest nonvanishing order term. In any case, what we want to capture are the Lyapunov exponents in the instability region. These are driven by the purely non-Hermitian part of the Hamiltonian, and could be obtained through the BCH formula. Actually, even the first order of the BCH formula, in most of the cases that we have examined, is sufficient to yield very accurate results. There are cases though, when considering periodic fluctuations of the frequency (for example, in the Mathieu equation, cf. Sec. III), where the second-order term is the first nonvanishing order term in the expansion of the BCH formula. This is actually the reason why it is necessary to include the second order term of BCH in our analysis.

At this point we should note that our method could also be used in the case where dissipation is present as well. In brief, the appearance of a dissipative factor $-2\gamma\dot{x}$ in the equation of motion will cause, as expected, an overall exponential decay of the energy of the form of $e^{-2\gamma t}$ that will compete any possible exponential growth of the energy due to frequency fluctuations; if the Lyapunov exponent L is greater than 2γ, the energy will still grow exponentially. A secondary effect of dissipation is to modify a bit the form of the Lyapunov exponent due to the introduction of an $\iota\gamma\sigma_3$ term in the anti-Hermitian part of the Hamiltonian H_A. Finally, the appearance of a γ factor will shift the characteristic frequency of the oscillator and, thus, this modified frequency will determine the resonances of the fluctuations (in case of periodic ones).

In the following sections we will illustrate, by a few examples of $g(t)$, the usefulness and the power of the above analytical formula. It should be noted that the parameter ω_0, in all formulas above, could be set equal to 1 by rescaling the time parameter as $t \to \omega_0 t$ and, thus, it will be omitted from now on.

III. MATHIEU EQUATION

A simple example, where the power of the approximate formula (15) shows up transparently, is the case of the Mathieu equation. In the following, we shall analyze the solution of the Mathieu equation

$$\frac{d^2y}{dx^2} + [a - 2q\cos(2x)]y = 0, \quad (19)$$

not by the usual power expanding technique (cf., Ref. [12]), but with our approximate formula.

Following the transformation $t=\sqrt{a}x$, the Mathieu equation takes the form

$$\frac{d^2y}{dt^2} + [1 - \epsilon\cos(\Omega t)]y = 0, \quad (20)$$

where $\epsilon \equiv 2q/a$ and $\Omega \equiv 2/\sqrt{a}$. This is exactly the form of a harmonic oscillator with a harmonically varying frequency around unity. Thus, in the language used to describe the oscillator in the previous section, $\omega_0=1$ and $g(t)=-(\epsilon/2)\times\cos(\Omega t)$.

Since $g(t)$ is a periodic function, correlations of order higher than two, we have omitted in our approximation, are not negligible. Therefore, we anticipate the faithfulness of the approximate formula (15) to become worse as time passes. On the other hand, the periodicity of $g(t)$ assures that, if we include in the propagator for one period of $g(t)$, that is

for $P=2\pi/\Omega$, the rotating phase that corresponds to this period,

$$U_2(0,P=2\pi/\Omega)=e^{-\imath\int_0^P H_{\mathrm{H}}\mathrm{d}t}U_2^{(I)}(0,P), \tag{21}$$

then we shall obtain a rather accurate estimate of the evolution of the energy of the system. Equation (21) gives the propagator in the Schrödinger picture and is obtained directly by inverting Eq. (6). Although, the inclusion of the rotating phase in the propagator seems unimportant due to its Hermitian character, it is actually essential since the successive operation of the non-Hermitian propagator interferes with the starting phase of the system either constructively or destructively, since the absorption of energy of the oscillator depends on its initial phase. Thus, in case of periodic fluctuations of the frequency, at each period one must supply the phase shift to get the overall evolution of the energy. The extra factor $\exp(-\imath\int_0^P H_H\mathrm{d}t)$ in Eq. (21) has a group-theoretic origin. It corresponds to the $e^{\imath ka}$ of Bloch electron band theory in a periodic potential. In the next section, where the stochastic case of $g(t)$ is explored, the relation between the phase of the system and the non-Hermitian propagator is further clarified.

The propagator of Eq. (21) gives the evolution of the system at any time, that is an integer multiple of the period, since it is repeated as it is at every period. The rotating phase part is

$$e^{-\imath\int_0^P H_{\mathrm{H}}(t)\mathrm{d}t}=\begin{pmatrix}\cos(2\pi/\Omega) & \sin(2\pi/\Omega)\\ -\sin(2\pi/\Omega) & \cos(2\pi/\Omega)\end{pmatrix}, \tag{22}$$

while the propagator $U_2^{(I)}$ is given in Eq. (17) with $T=P$. The integrals represented by the a_i's of Eqs. (18) cannot be computed analytically. For small values of ϵ, though, we can express the a_i's as power series of ϵ,

$$a_1=\epsilon\frac{\sin 4\pi/\Omega}{\Omega^2-4}+\frac{\epsilon^2}{8}\frac{\sin 4\pi/\Omega}{\Omega^2-1}+O(\epsilon^3), \tag{23}$$

$$a_3=2\epsilon\frac{\sin^2 2\pi/\Omega}{\Omega^2-4}+\frac{\epsilon^2}{4}\frac{\sin^2 2\pi/\Omega}{\Omega^2-1}+O(\epsilon^3), \tag{24}$$

$$a_2=-\frac{\epsilon^2}{2(\Omega^2-4)}\left[\frac{\pi}{\Omega}+\frac{2\sin 4\pi/\Omega}{\Omega^2-4}\right]+O(\epsilon^3). \tag{25}$$

Each order at the series expansion of a_1, which is of order ϵ^n, yields its maximum absolute value at frequency $\Omega=2/n$, where n is the order of the term itself. The corresponding order terms of a_3 are of one order of magnitude, with respect to the mismatch of frequency to $\Omega=2/n$, lower than the terms of a_1. Finally, the lowest order term of a_2 (the ϵ^2 term) are of order unity at each of these frequencies (even for $\Omega=2$ as one can easily check directly by expanding the term around that frequency). These special features of the above order terms, especially of a_1, around the frequencies $\Omega=2/n$, brings out directly the characteristic resonances of the Mathieu equation. On the other hand the nonvanishing ϵ^2 term of a_2 for all the resonant frequencies justifies the inclusion of the second-order term in the use of the BCH formula in our analysis in the first place. Moreover, our approximate formula gives correctly the Lyapunov exponents around these frequencies as well. The eigenvalues $\lambda_\pm$ of the propagator of the system $U_2(0,P)$ are related to the Lyapunov exponent of the system by

$$L=\frac{\log(\max\lambda_\pm)}{P}. \tag{26}$$

Here we are going to use our approximate propagator instead. The eigenvalues $\lambda_\pm$ of the approximate propagator U_2 are the roots of the binomial

$$\lambda^2-\lambda\mathcal{R}+\mathcal{D}=0, \tag{27}$$

where $\mathcal{R}$ is the trace of the propagator, and $\mathcal{D}$ is its determinant, here equal to 1 due to the special form of the two matrices of Eqs. (22) and (17). Thus, the Lyapunov exponent that our approximate formula yields for the system is

$$L=\frac{\log\left(|\mathcal{R}/2|+\sqrt{(\mathcal{R}/2)^2-1}\right)}{2\pi/\Omega}, \tag{28}$$

where $\mathcal{R}/2$ yields the following form for our propagator:

$$\frac{\mathcal{R}}{2}=\cos\left(\frac{2\pi}{\Omega}\right)\cosh a+\sin\left(\frac{2\pi}{\Omega}\right)a_2\frac{\sinh a}{a}. \tag{29}$$

Let us first focus our interest in the region of frequencies around the basic parametric resonance $\Omega=2+\delta$, where $|\delta|\ll 1$. By expanding all quantities involved to the lowest order with respect to ϵ and δ the expression for the Lyapunov exponent yields

$$L=\frac{1}{2}\sqrt{\left(\frac{\epsilon}{2}\right)^2-\delta^2}. \tag{30}$$

This is exactly what one gets from a straightforward analysis of the exact problem (see Ref. [13]). In Fig. 1 we have plotted a comparative diagram of the exact evolution of the Mathieu equation along with the one computed with our approximate formula (15), for a few characteristic parameter values. It is obvious from this graph that the approximate formula is very faithful in a much wider region than the one we have checked analytically $|\epsilon|,|\delta|\ll 1$ around the parametric resonance frequency $\Omega=2$. This was more than it was anticipated since the correlations of order higher than two of the Mathieu equation are not negligible and the approximate formula was built under the assumption that higher order correlations are of no importance. However, the dependence of $\phi(t)$ on $g(t)$ [cf., Eq. (11)] brings contributions of higher order correlations at lower order terms; this justifies the great power of our approximation.

Next we test the faithfulness of our approximate formula in the neighborhood of the second parametric resonance, $\Omega=1+\delta$, with $|\delta|\ll 1$. Based on the periodicity of the propagator of the Mathieu equation, it could be shown (see Refs. [14,15]), by expanding δ as $\delta_0+\delta_1\epsilon+\delta_2\epsilon^2+\cdots$ and searching for marginally periodic solutions in the form of powers of ϵ: $y_0(t)+\epsilon y_1(t)+\epsilon^2 y_2(t)+\cdots$, that the boundary lines that separate stable and unstable solutions of Mathieu equation is given by

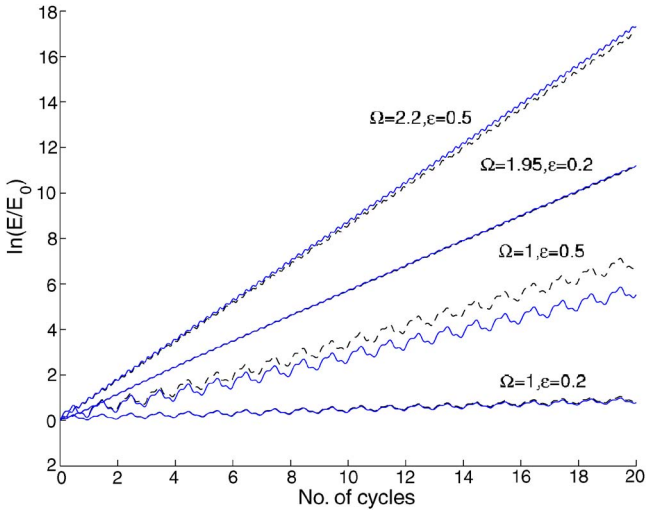

FIG. 1. (Color online) Comparison between the exact time evolution of the Mathieu equation and the analytical evolution predicted by the approximate formula of Eq. (15). There are four pairs of curves. For every pair the (blue) solid line shows the exact evolution, while the (black) dashed one shows the approximate evolution. All graphs depict the evolution of the energy of the system for a total time that corresponds to 20 cycles of the periodic term in Eq. (20). All cases shown here correspond to such values of parameters Ω, ϵ that the energy grows exponentially (within an unstable region). Only cases around the first two resonant frequencies ($\Omega=2$ and $\Omega=1$) are shown here. For frequencies around $\Omega=2$ the approximate solutions (dotted curves) match almost perfectly the corresponding exact solutions (solid curves), even at values of ϵ as large as 0.5! The match is not so perfect for the higher order frequencies at large values of ϵ ($\epsilon=0.5$) but it is still very good as long as ϵ is smaller than 0.2. It should be noted that the approximate solutions plotted here have been computed by alternating the rotating phase matrix of Eq. (22) and the propagator $U_2^{(I)}$ of Eq. (17) at every period $2\pi/\omega$.

$$\delta_{0+}^{(\Omega\simeq 1)} = \delta_{1+}^{(\Omega\simeq 1)} = 0, \quad \delta_{2+}^{(\Omega\simeq 1)} = \frac{1}{24},$$

$$\delta_{0-}^{(\Omega\simeq 1)} = \delta_{1-}^{(\Omega\simeq 1)} = 0, \quad \delta_{2-}^{(\Omega\simeq 1)} = -\frac{5}{24}. \tag{31}$$

Of course the method mentioned above could be used to determine the unstable region around any resonant frequency, $\Omega=2/n$. However, our approximate analytical formula (15) not only leads to exactly the same boundaries for the unstable region of this resonant frequency, but it yields the corresponding Lyapunov exponent as well:

$$L = \sqrt{-\delta^2 - \frac{\delta\epsilon^2}{6} + \frac{5\epsilon^4}{16\cdot 36}} = \sqrt{\left(\frac{\epsilon^2}{24} - \delta\right)\left(\frac{5\epsilon^2}{24} + \delta\right)}. \tag{32}$$

The expression above is given at the lowest order of ϵ, and δ. In the series expansion of a_1, a_3, a_2 we have kept only terms that are of order ϵ^2, which is the lowest order showing up, and in the series expansion of $\cos(2\pi/\Omega)$ and $\sin(2\pi/\Omega)$ we have kept only the first nonconstant terms. Note that, while for $n=1$, the term δ is of the same order as ϵ [see Eq. (30)], for $n=2,3,\ldots,\delta$ is of higher order than ϵ (cf. Figs. 20.8–20.10 of Ref. [12]).

The examination of the first two resonance bands, $\Omega\simeq 2$ and $\Omega\simeq 1$, suggest that our approximate analytical formula describes very faithfully the evolution of the Mathieu equation in the region of every resonant frequency, at least for low values of ϵ.

Another periodic case that is of special physical importance is the case of a periodic two-valued frequency change around a central value ω_0^2. Imagine that the frequency of the oscillator is given by

$$\omega^2 = \omega_0^2 + \lambda \times \begin{cases} -\dfrac{1}{T} & \text{if } (n-1)T < t < nT - \epsilon \\ \left(\dfrac{1}{\epsilon} - \dfrac{1}{T}\right) & \text{if } nT - \epsilon \leqslant t \leqslant nT, \end{cases} \tag{33}$$

where the three parameters λ, ϵ, and T control the strength, the width of the pulses, and the period of the frequency fluctuations, respectively, and n is any integer number. A potential with periodic spatial dependence (instead of temporal dependence) like that describes the so-called Kronig-Penney model of a periodic lattice. Actually, the Schrödinger equation of such a potential has exactly the same form of our harmonic oscillator equation of motion, with time variable t being replaced by the spatial variable x. Thus we could use this well-known quantum-mechanical problem to check once again our approximative formula. We will specialize our investigation in the case of small λ values where our approximate method is to be trusted, and a short ϵ time-parameter so that the pulses of large frequency shifts could be considered as delta functions (for which case analytical results could be derived for the Kronig-Penney potential).

The systematic analysis of the Kronig-Penney potential in the latter case ends up in a transcendental equation for the determination of the allowed energy levels of the system. The gaps between them give the forbidden energy bands which are restricted by the following λ-dependent intervals:

$$\frac{2}{n}\left(1 - \frac{\lambda}{2\pi n\omega_0}\right) \leqslant \frac{2\pi}{T\omega_0} \leqslant \frac{2}{n}\left(1 + \frac{\lambda}{2\pi n\omega_0}\right), \tag{34}$$

at first order with respect to λ (and for $\epsilon \ll T$). The n is any positive integer number that characterizes the order of the resonant frequency. Following exactly the same procedure as with the Mathieu case, we find that at the lowest order with respect to λ and ϵ the Lyapunov exponent for the energy of this periodic problem is given by

$$L = \sqrt{\left(\frac{\lambda}{2\omega_0 T}\right)^2 - \left(\frac{n\pi}{T} - \omega_0\right)^2}. \tag{35}$$

The term inside the last parentheses measures the deviation of the frequency of the fluctuations from the $2\omega_0/n$ resonant frequencies. The estimated Lyapunov exponent becomes zero at exactly the boundaries of the "forbidden bands" that were shown above. Thus, once again our approximate method not only gives the exact analytical result of the

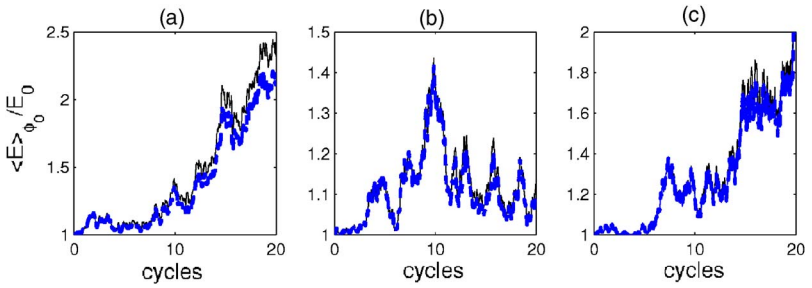

FIG. 2. (Color online) In this diagram we test the faithfulness of our approximate formula [Eq. (38)] for the energy evolution of the system averaged over its initial phase, for three different realizations of stochastic fluctuations of its frequency parameter ω^2. In all cases the (black) solid curve represents the exact average energy vs time, while the (blue) dashed one depicts the output of Eq. (38) computed for the corresponding stochastic realization. Diagram (a) corresponds to a fractional Gaussian stochastic variable with Hurst parameter 0.85, diagram (b) corresponds to an autoregressive stochastic variable with parameter a=0.80, and diagram (c) corresponds to a random telegraphic stochastic variable with average pulse length equal to ten steps. More information about the three kinds of stochastic variables considered are presented in the text. All cases considered have been normalized to have the same standard deviation $\sigma_{\omega^2}=0.2$.

boundaries themselves, but it also provides an analytical expression of the Lyapunov exponent within the whole region of these bands. In the analysis that leads to the results given in Eq. (35) we have used the Kronig-Penney potential with a small but definite ϵ value, since the use of delta functions instead would lead to ill-defined mathematical formulation of the approximate analysis.

IV. NOISE INDUCED PARAMETRIC RESONANCE

Now, based again on the approximate formula of Eq. (15), we will examine what kind of stochastic functions $g(t)$, describe noisy fluctuations of the parameters of the dynamical system, will lead to exponential growth of the energy of the system. The energy of the system at time T will be given by

$$E(T) = \langle \Psi_0 | (U_2^{(I)}(T))^\dagger U_2^{(I)}(T) | \Psi_0 \rangle. \tag{36}$$

The Hermitian part of the Hamiltonian, as was discussed above, does not change the energy, therefore there is no difference in using the $U_2^{(I)}$ or the U_2 propagator. In the previous section, in the analysis of the Mathieu equation, the periodicity of $g(t)$ gave us the advantage of computing everything up to one period of the frequency parameter, and then repeat the same propagator for the subsequent periods. In that case, the inclusion of the rotating phase, that relates the U_2 with the $U_2^{(I)}$, was essential, since the effect of the periodically varying frequency on the system depends on the phase difference between the varying frequency and the system itself. For the accuracy achieved in the Mathieu case it was important to include the rotating phase after each period. Now, the stochastic character of $g(t)$ lacks periodicity and makes it useless to include any rotating phase in the expression for the energy evolution.

If we substitute the components of the matrices in Eq. (36) we get

$$E = \cosh(2a) + 2a_2^2\left(\frac{\sinh a}{a}\right)^2 + \cos(2\phi_0 - \theta(a_1,a_3,a_2)) \times \frac{2\sinh a}{a}\sqrt{(a_1^2+a_3^2)\left(\cosh^2 a + a_2^2\frac{\sinh^2 a}{a^2}\right)}, \tag{37}$$

where the a_1, a_2, a_3, a are the integrals given at the end of Sec. II, and ϕ_0 is the initial phase of the oscillator, while $\theta(a_1,a_2,a_3)$ is a complicated function of a_i's. This expression shows that the average value of the energy $\langle E \rangle_{\phi_0}$, over all initial phases, is given by the first two terms which are greater than (or equal to) 1, and they will grow larger and larger with time if a is a monotonic growing function of time,

$$\langle E \rangle_{\phi_0} = \cosh(2a) + 2a_2^2\left(\frac{\sinh a}{a}\right)^2. \tag{38}$$

Moreover, if we neglect the a_2 terms, which are of higher order with respect to the magnitude of $g(t)$, compared to the a_1, a_3 terms, then $\langle E \rangle_{\phi_0}$, given by Eq. (38), simplifies to

$$\langle E \rangle_{\phi_0} = \cosh(2a), \tag{39}$$

where

$$a^2(t) = \int_0^t \int_0^t dt_1 dt_2 g(t_1) g(t_2) \cos\left(2(t_1 - t_2) + 2\int_{t_2}^{t_1} g(t')dt'\right). \tag{40}$$

Before proceeding to use the above formula to make predictions about the statistics of the evolution of the energy, we compare the evolution described by this simple integral versus the real evolution of the energy of the system to test the validity of our approximation. Stochastic fluctuations of quite large magnitude (with variance $\sigma_{\omega^2}=0.2$) lead to almost unnoticeable discrepancies, even for long times compared to the oscillator's period (see Fig. 2). This justifies the various approximations used up to this point [that is neglecting the higher than third order of correlations, and omitting the a_2 terms in Eq. (38)]. We have actually tested the significance of the a_2 terms in our final expression for the average energy, and it was always so tiny that you could not discern the corresponding curve from the one with that term omitted.

Now that we have an analytical formula in our hand, we could get an estimate for an ensemble average evolution of the energy of the system under a stochastic fluctuation of its frequency with specific characteristics. Looking for the low-

est order, with respect to g, expressions of the average (over initial phase) time evolution of the energy, we could omit the integral of $g(t)$ inside the cosine term of the formula above. In the following subsections we show that this approximation is valid for sufficiently small magnitude of fluctuations of $g(t)$. What remains is a simple integral whose value depends on the statistics of $g(t_1)g(t_2)$. For stationary stochastic functions the integrand is simply a function of t_1-t_2, since for stationary stochastic functions $\langle g(t_1)g(t_2)\rangle=\langle g((t_1+t_2)/2+(t_1-t_2)/2)g((t_1+t_2)/2-(t_1-t_2)/2)\rangle$ does not depend on $(t_1+t_2)/2$ and is an even function of t_1-t_2:

$$\langle g(t_1)g(t_2)\rangle_{\text{ensemble}} = A(|t_1 - t_2|). \tag{41}$$

Under these quite general assumptions

$$\langle a^2(t)\rangle_{\text{ensemble}} \simeq 2\int_0^t ds \int_{s/2}^{t-s/2} \mathrm{d}x A(s)\cos(2s)$$
$$= 2\int_0^t ds(t-s)A(s)\cos(2s). \tag{42}$$

Thus, we are led to the following simple prediction:

$$\left\langle \left(\frac{\cosh^{-1}\langle E\rangle_{\phi_0}}{2} \right)^2 \right\rangle_{\text{ensemble}} \simeq 2\int_0^t dsA(s)\cos(2s)(t-s)$$
$$= F(t). \tag{43}$$

At this point we should point out that for stochastic functions that are Gaussian, the omitted term in the argument of cosine of Eq. (40) could be actually taken into account in the averaging procedure and thus one could use the expression given by Eq. (42) with a slightly different function for $A(s)$ from the one given in Eq. (41):

$$A(s) = \exp(-2\langle\Psi^2\rangle)\left[\langle g(0)g(s)\rangle - 4\langle g(0)\Psi\rangle\langle g(s)\Psi\rangle\right], \tag{44}$$

where $\Psi \equiv \Psi(0,s)=\int_0^s g(t)\mathrm{d}t$ (for the derivation see Appendix B). The new function is still an even function. We have confirmed, though, that practically it does not yield any essential modifications to the results.

The function $F(t)$ depends on the kind of stochastic function assumed, serves as a bulk estimate of the time-evolution of the average energy. The exact evolution of the average energy itself is very difficult to calculate since we then need to know all the moments of g besides the second one, in order to compute the average value of the hyperbolic cosine of this quantity. Furthermore, such information is not of great use since the variation of the energy grows usually even faster than the average energy itself. In spite of this, the concavity of the function $f(x)=\cosh(2\sqrt{x})$ [that is $f''(x)>0$], ensures that

$$\langle f(x)\rangle > f(\langle x\rangle). \tag{45}$$

Thus, we are ensured that the growth rate of the average energy of the system is at least as large as the following simple estimate:

$$\langle E\rangle_{\phi_0,\text{ensemble}} > \cosh(2\sqrt{F(t)}). \tag{46}$$

Therefore, generally, stochastic fluctuations of the dynamical parameters of the system that are characterized by a function $F(t)$ of the form of Eq. (43), that has an overall positive rate with time, will lead, on average, to exponential increase of the energy of the system.

In order to illustrate the above statement with a few examples, we have constructed a set of long data series of stochastic variables of various types, with given specific parameters, that characterize each specific type of stochastic variables. We then used them to evolve numerically the harmonic oscillator with an exact propagator assuming that the square of its frequency ω^2 is varying discontinuously and according to the successive values of each data set in time steps τ. The assumed time steps τ were small with respect to the unperturbed period of the oscillator T ($\tau \simeq T/200$), while in all cases studied the variance of the stochastic variable was $\sigma_{\omega^2}=0.2$. Then the evolution of the energy of the oscillator was monitored for a large number of initial phases of the oscillator, and for a large number of noise realizations, of the same type. The average $\langle(\cosh^{-1}\langle E\rangle_{\phi_0}/2)^2\rangle_{\text{ensemble}}$ was recorded for a few periods of the oscillator, and finally, it was compared with the corresponding analytical form given by Eq. (42). In the following examples the accuracy of the approximation of our method and its power as an analytical forecast of the evolution of the energy of the system is presented.

Let us use as a first example of $g(t)$ a fractional Gaussian noise (FGN) variable (see Ref. [16]) with variance σ and Hurst parameter H ($0<H<1$). The FGN's autocorrelation function is given in Appendix C, along with other useful expectation values for that kind of Gaussian stochastic variable. From them we could write an expression for $F(t)$ in terms of a sum over integers (see Appendix B). In Fig. 3 we have plotted the numerically computed $F(t)$ alongside the numerically computed ensemble average of $(1/4)\times(\cosh^{-1}\langle E\rangle_{\phi_0})^2$ for a few Hurst parameters. The accuracy is quite good, although it seems to be deteriorating with time. However, this is mainly caused by the fact that statistical dispersion grows exponentially with time as well and, hence, the statistical expectation values should not be trusted a lot at late times. The comparative diagrams suggest that one could rely on the analytical forms for $F(t)$ to get a gross estimate of the Lyapunov exponent of the energy of the system.

As a second example we have used a stationary autoregressive (AR) variable (see Ref. [17]) with variance σ and parameter a ($|a|<1$). In this case, again, the comparison between analytical and statistical numerical results that are plotted in Fig. 4 are quite satisfactory. Predictions for the average energy evolution could be based on our analytical formula for $F(t)$. It should be pointed out again that the comparative diagrams for the cases of FGN and AR, which are shown in Figs. 3 and 4, are based on the approximation of negligible contribution of the integral term inside the cosine

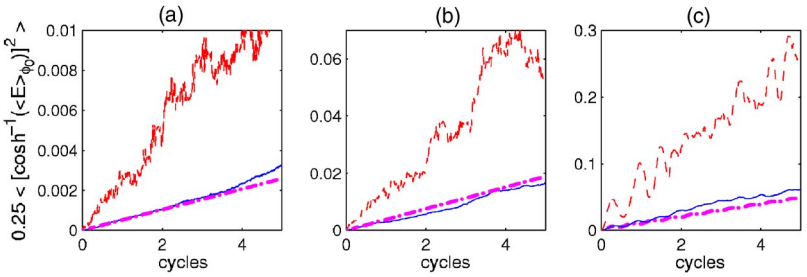

FIG. 3. (Color online) Three diagrams (computed from 100 FGN realizations) that depict the average energy growth of the system when its frequency fluctuates according to an FGN variable with Hurst parameter (a) $H=0.3$, (b) $H=0.6$, and (c) $H=0.9$. In all cases, the (red) dashed highly fluctuating curve at the top represents, at each time moment, the maximum energy of the system among all the FGN realizations used; this is indicative of the variance of the average value. The remaining two curves represent the analytic function $F(t)$ (solid thin blue curve), and the ensemble average value for the quantity $(\cosh^{-1}\langle E\rangle_{\phi_0})^2/4$ (thick dot-dashed magenta curve), which is a bold estimate for one fourth of the square of the Lyapunov exponent for the energy. The agreement between the theoretical prediction and the actual ensemble average is moderately good; moreover, the theoretical prediction reproduces the wiggly fine structure along the evolution that is present at high values of the parameter H, as well.

argument of Eq. (40), although a much more accurate—though complicated—formula could be written for the average evolution of the energy in case of Gaussian stochastic variables (cf., Appendix C).

Finally, a non-Gaussian stochastic variable has been tested against our approximate estimate, namely the case of a random telegraphic signal (RTS) has been used to describe the fluctuations of the oscillator's frequency. The magnitude of the square pulse is assumed to be σ while the characteristic time τ_0 for a pulse is assumed to be a few tens of numerical time steps τ, which is the parameter adjusting the rate at which all our noise variables are assumed to vary. As we said previously, since this kind of noise is not Gaussian we cannot compute the expectation value of the whole integrand of Eq. (40). The best we can do to get an analytical expression that approximately describes the average evolution of the energy is to neglect the g-dependence of the cosine term and use the expectation value of the remaining expression. Once again the predicted average compares very well with the actual average, as shown in Fig. 5.

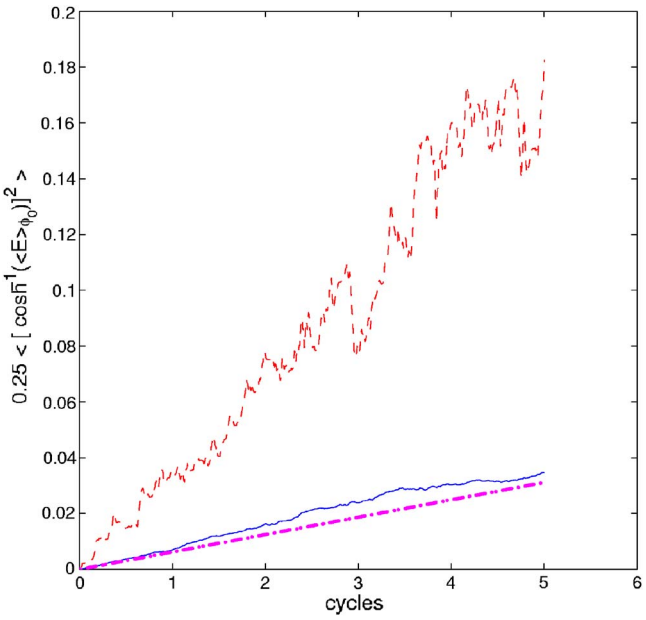

FIG. 4. (Color online) A comparative diagram, like the ones of Fig. 3, for the case of an autoregressive type of stochastic fluctuations of the oscillator's frequency with parameter a=0.5. The solid thin (blue) line represents the analytical approximate evolution, while the thick dot-dashed (magenta) curve represents the average evolution of 100 noise realizations.

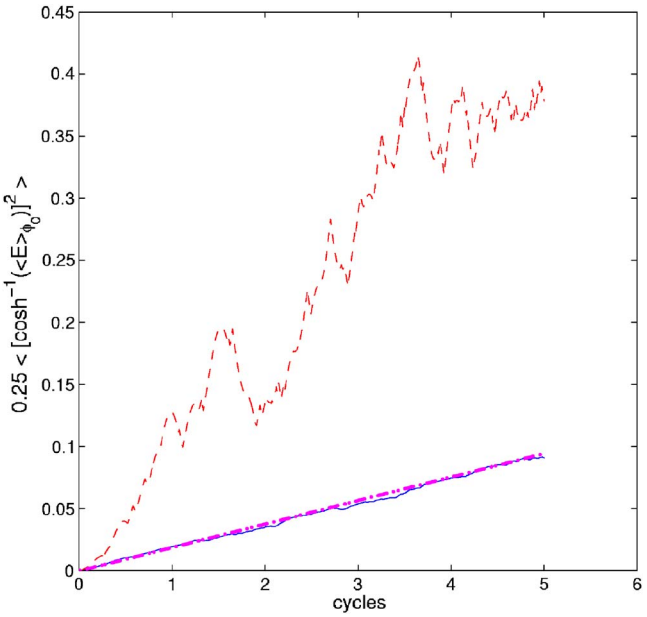

FIG. 5. (Color online) A comparative diagram, like the ones of Fig. 3, for the case of a random telegraphic square signal that describes the fluctuations of the oscillator's frequency. The average length of the square pulses is assumed to be ten times the characteristic time step used in frequency renewal. The solid thin (blue) line represents the analytical approximate evolution, while the thick dot-dashed (magenta) curve represents the average evolution of 100 noise realizations.

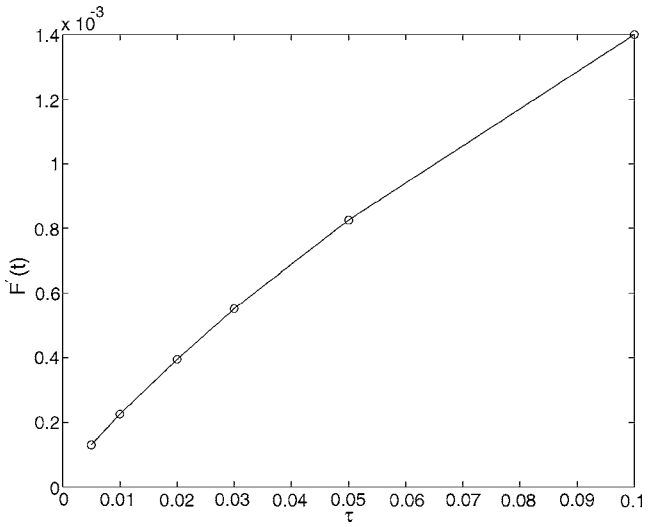

FIG. 6. The value of dF/dt [for $F(t)$ see Eq. (43)] has been drawn for the case of FGN fluctuations with $H=0.6$, at some sufficiently large values of t where $F(t)$ is almost a linear function of t vs the noise time renewal parameter τ. As τ goes to zero the slope of $F(t)$ goes to zero as well, but not in a linear way, due to τ-dependence of the summed up quantities in Eq. (C4).

Finally, we could use our analytical expressions to investigate what is the effect of the characteristic time step at which the fluctuations occur to the overall exponent that the energy of the system evolves. From Eq. (C4), which describes how the integral that yields the function $F(t)$ should be computed, it is clear that for sufficiently fast decreasing $A(k)$ (which was always the case in all the examples that we have explored), as τ goes to zero it leads to a vanishing $F(t)$ for any given time t, although not in a linear manner due to the complicated τ-dependence of the sum itself. This nonlinear τ behavior is depicted in Fig. 6.

V. GENERAL CONCLUSIONS

In this paper, we have developed a formalism of quantum mechanical "inspiration" to describe the time evolution of a classical oscillatory system with fluctuating parameters. By an appropriate choice of the quantum mechanical interaction picture representation, we have obtained an expression for the Lyapunov exponent of the energy evolution of the dynamical system. Our approach proved to be very powerful in handling even highly correlated processes, such as in the case of the Mathieu equation or the quantum mechanical problem of the Kronig-Penney potential in solid state physics. We made it possible to determine not only the unstable region around any resonant frequency $\omega=2/n$, but gave analytical expressions for the Lyapunov exponent as well [cf. Eqs. (30) and (32)]. Also, in the case of Gaussian fluctuations of the parameters, we could derive analytical expressions not only for the approximate form of Eq. (40), when the $g(t)$ term in the argument of cosine has been omitted, but also for the whole form, showing directly that our approximation in neglecting the $g(t)$ term is legitimate. However, the expression (44) may prove useful elsewhere in the theory of stochastic functions. Our method proved very good for all other non-Gaussian cases treated. The approach lends itself to generalizations for use in a wide field of applications.

ACKNOWLEDGMENTS

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APPENDIX A: COMPUTING THE TIME ORDER PRODUCT

Here we present, in a more scholastic manner, the computations that led to the final expression for the propagator of the oscillator in the interaction picture, given by Eq. (15). Let us start from ordering the terms of two successive propagators of infinitesimal duration δ, using the Baker-Campbell-Hausdorff formula to express them as a single exponential

$$\begin{aligned} U(t_k,t_k+\delta)U(t_{k-1},t_{k-1}+\delta) &= e^{-\iota\delta H^{(I)}(t_k)}e^{-\iota\delta H^{(I)}(t_{k-1})} \\ &= \exp\Big(-\iota\delta(H^{(I)}(t_k)+H^{(I)}(t_{k-1})) \\ &\quad -\frac{\delta^2}{2}[H^{(I)}(t_k),H^{(I)}(t_{k-1})]+O(\delta^3)\Big). \end{aligned} \tag{A1}$$

Now, by writing down the explicit form of the interaction Hamiltonian [see Eq. (10)] it is straightforward to compute our two-step time-ordered propagator

$$\begin{aligned} U(t_{k-1},t_{k+1}) = \exp[\,&\delta\omega_0(-(g_kc_k+g_{k-1}c_{k-1})\sigma_1 \\ &+(g_ks_k+g_{k-1}s_{k-1})\sigma_3) \\ &-\delta^2\omega_0^2g_kg_{k-1}(s_kc_{k-1}-c_ks_{k-1})\iota\sigma_2+O(\delta^3)], \end{aligned} \tag{A2}$$

where the indices refer to discrete time moments, e.g., $g_k=g(t_k)$, while the c's and s's are abbreviations for $\cos 2\phi$ and $\sin 2\phi$, respectively. Next our aim is to use this ordering with respect to the infinitesimal time δ to construct a single exponential expression for the whole finite-time propagator $U(0,T)$. Although even at this point, with only two successive propagator terms, it is rather obvious what will be the general form, we will compute the product of three successive such terms to make it more clear. Indeed

$$U(t_{k-1},t_{k+2}) = U(t_{k+1},t_{k+2})U(t_{k-1},t_{k+1}) = \exp[\delta\omega_0 g_{k+1}(-c_{k+1}\sigma_1 + s_{k+1}\sigma_3)]\exp[\delta\omega_0(-(g_k c_k + g_{k-1}c_{k-1})\sigma_1 + (g_k s_k + g_{k-1}s_{k-1})\sigma_3) - \delta^2\omega_0^2 g_k g_{k-1}(s_k c_{k-1} - c_k s_{k-1})\iota\sigma_2 + O(\delta^3)]$$
$$= \exp[\delta\omega_0(-(g_{k+1}c_{k+1} + g_k c_k + g_{k-1}c_{k-1})\sigma_1 + (g_{k+1}s_{k+1} + g_k s_k + g_{k-1}s_{k-1})\sigma_3) + \delta^2\omega_0^2(g_{k+1}c_{k+1}g_k s_k + g_{k+1}c_{k+1}g_{k-1}s_{k-1} + g_k c_k g_{k-1}s_{k-1} - g_{k+1}s_{k+1}g_k c_k - g_{k+1}s_{k+1}g_{k-1}c_{k-1} - g_k s_k g_{k-1}c_{k-1})\iota\sigma_2 + O(\delta^3)]. \quad \text{(A3)}$$

The above computation is straightforward and leads directly to the general formula given in the text when one takes the limit $\delta\to 0, N\to\infty$. The actual expression of Eq. (15) is approximate since all terms arising from the δ^3 order terms have been omitted. This is based on the assumption that $g(t)$ is of small magnitude, and that correlations of $g(t)$ of order higher than two are negligible. The types of $g(t)$'s that have been used to check this approximate formula (cf., Secs. III and IV) have verified that our approximation is quite accurate.

APPENDIX B: COMPUTING THE AVERAGE EVOLUTION OF THE ENERGY DUE TO A GAUSSIAN STOCHASTIC NOISE

For a zero-mean Gaussian variable, X, with variance, σ^2, the expectation value (average over ensemble) of $e^{\iota X}$ is (cf., Ref. [12])

$$\langle e^{\iota X}\rangle = e^{-\sigma^2/2}. \quad \text{(B1)}$$

Since the above expression holds for any zero-mean Gaussian variable, the expectation value of the actual function that appears in the integral of Eq. (40) could be easily computed from the statistics of suitable derivatives of the function

$$f(t_1,t_2) = e^{\iota(2(t_1-t_2)+\lambda_1 X_1+\lambda_2 X_2+\ldots+\lambda_n X_n)}, \quad \text{(B2)}$$

where $\lambda_1,\lambda_2,\ldots,\lambda_n$ should all be set equal to 2τ at the end, and $X_i = g(t_2+(i-1)\tau)$, while τ is the physical time step used for the renewal of noise. In this notation $X_1 = g(t_2)$ and $X_n = g(t_1)$. The noise function is assumed to be given as a series of numbers that change discontinuously at the integer multiples of τ. Thus, $g(t)$ is assumed to operate physically as

$$g(t) = g([t/\tau]), \quad \text{(B3)}$$

where $[a]$ denotes the integer value of a, and τ is the time step by which the noise function is assumed to renew. The dependence of our conclusions on the magnitude of the time step τ have been investigated in Sec. IV. Hence,

$$g(t_1)g(t_2)\cos\left(2(t_1-t_2) + 2\int_{t_2}^{t_1} g(t)\mathrm{d}t\right) = -\Re\left(\frac{\partial^2}{\partial\lambda_1\,\partial\lambda_n} f(t_1,t_2)\right)_{\lambda_1=\cdots=\lambda_n=2\tau}, \quad \text{(B4)}$$

where $\Re(\cdots)$ denotes the real part of $(\cdots)$. However, the expectation value of f could be obtained since the X_i's are zero-mean Gaussian variables. Thus

$$\langle f(t_1,t_2)\rangle = e^{2i(t_1-t_2)}\exp\left(-\frac{1}{2}\langle(\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n)^2\rangle\right). \quad \text{(B5)}$$

Finally, by interchanging the order of the real value operation and the derivatives with respect to λ's, and the computation of the expectation value, we get

$$\left\langle g(t_1)g(t_2)\cos\left(2(t_1-t_2) + 2\int_{t_2}^{t_1} g(t)\mathrm{d}t\right)\right\rangle = \cos(2(t_2-t_1))\exp(-2\langle\Psi^2\rangle) \times[\langle g(t_1)g(t_2)\rangle - 4\langle g(t_1)\Psi\rangle\langle g(t_2)\Psi\rangle], \quad \text{(B6)}$$

where $\Psi \equiv \Psi(t_2,t_1) = \int_{t_2}^{t_1} g(t)\mathrm{d}t$. Now, this is the general formula one should use to evaluate the average time evolution of the energy of the system, through Eq. (40), for any kind of zero-mean Gaussian variables. These should then be compared to numerical averages to test the predictability of our analytical formulas. As we have shown in the main text, the output of the comparison was satisfactory in all cases, even when the g-dependent argument of cosine in Eq. (40) was omitted. It remains to show that taking into consideration the nonzero Ψ terms of Eq. (B6) leads simply to minor corrections of the energy evolution that are still in good agreement with the numerical examples, see Fig. 7. The result of Eq. (B6) is therefore not of particular practical significance, and has been given mainly for completion.

As one can tell from the form of Eq. (B6) the function that has to be integrated over t_1,t_2 to get the energy evolution of the system depends again only on the difference t_1-t_2 and not on the actual values of t_1 and t_2. This was to be expected due to the stationarity of $g(t)$. Therefore we could simplify the corresponding double integral of Eq. (40) into the single integral

$$\langle a^2(t)\rangle = 2\int_0^t ds(t-s)A(s)\cos(2s), \quad \text{(B7)}$$

as in the approximate case assumed in the main text [cf., Eq. (42)], but with a somewhat more complicated function for $A(s)$:

$$A(s) \equiv \exp(-2\langle\Psi(s)^2\rangle)[\langle g(0)g(s)\rangle - 4(\langle g(0)\Psi(s)\rangle)^2], \quad \text{(B8)}$$

with $\Psi = \int_0^s g(x)dx$, which is again an even function, whatever kind of Gaussian variable is assumed for $g(t)$. Note that we

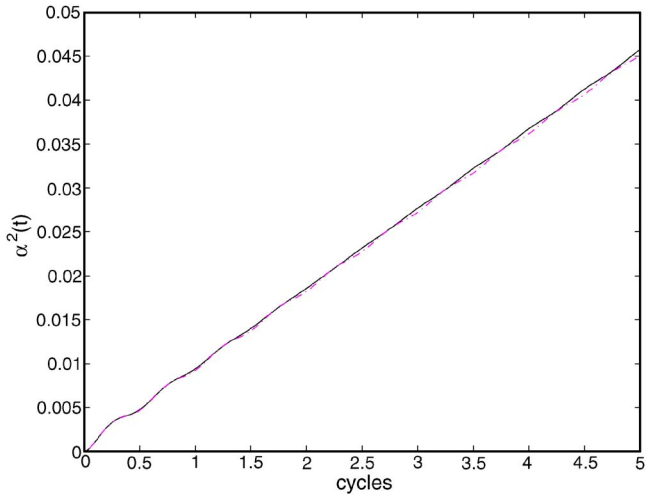

FIG. 7. (Color online) Comparison between the full Eq. (B6) (dot-dashed magenta curve) and the approximate one (solid blue curve) that we used in the main text [the same formula (B6) but with $\Psi=0$], for the case of FGN fluctuations with $H=0.8$ and $\sigma_{\omega^2}=0.2$. It is clear that the new complicated terms are of minor significance and they do not alter our conclusions. The same very good agreement between the accurate analytical expression for $a^2(t)$ and the approximate one, where the integral in the argument of cosine in Eq. (B6) has been omitted, shows up in all cases of Gaussian fluctuations considered here, independently of the values of its parameters.

have used the property that $\langle g(t_1)\Psi(t_2,t_1)\rangle=\langle g(t_2)\Psi(t_2,t_1)\rangle$. As was pointed out in the previous paragraph, the simpler form of function $A(s)$ used in the main text is very accurately describing the evolution of the system. In other words the g-dependence in the phase term of Eq. (40) is really insignificant, at least for the cases we have examined where σ_{ω^2} does not exceed 0.4. As a matter of fact, even if the more complicated form becomes important at higher variation values, it is not really useful since then we could not trust anymore the approximation scheme itself that is described in this paper. At least, from tests we have performed with high σ values ($\sigma_{\omega^2}=0.6$) the contribution of the new terms in $A(s)$ "push" the energy-evolution curve towards the right direction with respect to the numerical averages of the exact evolution behavior, even though the agreement between the exact average and our approximate estimate deteriorates as was expected to happen with large amplitude fluctuations.

APPENDIX C: USEFUL EXPECTATION VALUES FOR FGN'S AND AR'S

First, we show how the integral that gives the average energy evolution according to Eq. (42) should be computed numerically. Remember that the functions used here for the stochastic variable are discontinuous functions and thus the integrals could be written alternatively as discrete sums. Various parts of the explicit form of $A(s)$ in Eq. (B8) (for the case of a Gaussian stochastic variable) could then be written as sums of the autocorrelation function of the stochastic variable assumed. More specifically, if we define the autocorrelation function as

$$r(s) \equiv \langle g(0)g(s)\rangle = r([|s/\tau|]) = r(k), \tag{C1}$$

where $k=[|s/\tau|]$ is the integer part of $|s/\tau|$, then

$$\langle \Psi^2(s)\rangle = \begin{cases} \tau^2\left(kr(0) + 2\sum_{i=1}^{k-1}(k-i)r(i)\right) & \text{if } k \geqslant 1, \\ 0 & \text{if } k=0, \end{cases} \tag{C2}$$

and

$$\langle \Psi(s)g(0)\rangle = \begin{cases} \tau\sum_{i=0}^{k-1} r(i) & \text{if } k \geqslant 1, \\ 0 & \text{if } k=0. \end{cases} \tag{C3}$$

The reason that the sums are computed up to $k-1$, instead of k, is that until the kth renewal time ($t_1=t_2+k\tau$) the integral $\int_{t_2}^{t_1} g(t)dt$ is simply $\tau(g(t_2)+g(t_2+\tau)+\cdots g(t_2)+(k-1)\tau)$. Finally, the following discrete analogue for the single integral of Eq. (42) should be used:

$$F(t) = 2\tau\sum_{k=0}^{[t/\tau]}(t-k\tau)\cos(2k\tau)\begin{cases} A(k) & \text{if } k \geqslant 1 \\ \frac{1}{2}A(0) & \text{if } k=0 \end{cases}. \tag{C4}$$

The $1/2$ correction in the $A(0)$ terms reflects the fact that when we use the symmetry of $A(s)=A(-s)$ we take the double of the integral from 0 to k (this is where the overall 2 comes from), but then we count twice the values arising from $A(0)$.

The case of non Gaussian variables (e.g., the RTS example of Sec. IV), where we cannot write an analytic formula for the expectation value of the whole integrand of Eq. (40), and thus we rely on the approximate formula that omits the integral in the phase part (which is anyway practically negligible, as we have remarked earlier, in all cases considered), could be treated by exactly the same procedure as before by simply replacing $A(k)$ with $r(k)$, as in the main text.

Second, we give the analytic results of the various sums that $A(k)$ consists of in all three cases of stochastic variables used in the main text, namely, (1) fractional Gaussian noise (FGN), (2) autoregressive noise (AR), and (3) random telegraphic signal (RTS).

(1) For FGN Gaussian noises with variation σ^2 and Hurst parameter H, the discrete expectation values mentioned above are given by the following formulas

$$r(k) = \langle g(0)g(k)\rangle = \frac{\sigma^2}{2}((k+1)^{2H} + |k-1|^{2H} - 2k^{2H}), \tag{C5}$$

$$\langle g(0)\Psi(k)\rangle = \langle g(k)\Psi(k)\rangle = \frac{\sigma^2\tau}{2}(k^{2H} - |k-1|^{2H} + 1), \tag{C6}$$

$$\langle \Psi^2 \rangle = \tau^2 \sigma^2 k^{2H}, \tag{C7}$$

where

$$k = \left[\frac{t_1 - t_2}{\tau} \right]. \tag{C8}$$

The first formula above comes from the definition of this type of Gaussian noise (see Ref. [18]), while the rest are computed from the corresponding sums.

(2) For stationary AR Gaussian noises with parameter a ($|a| < 1$) the corresponding formulas are

$$\langle g(0)g(k) \rangle = \sigma^2 a^k, \tag{C9}$$

$$\langle g(0)\Psi(k) \rangle = \langle g(k)\Psi(k) \rangle = \tau\sigma^2 \frac{1 - a^k}{1 - a}, \tag{C10}$$

$$\langle \Psi^2 \rangle = \frac{\tau^2 \sigma^2}{(1 - a)^2} (k(1 - a^2) - 2a(1 - a^k)), \tag{C11}$$

where, again,

$$k = \left[\frac{t_1 - t_2}{\tau} \right]. \tag{C12}$$

Again the formulas above have been computed from the known correlation function of AR's, Eq. (C9) (see Ref. [17]).

(3) For the RTS signals which are not Gaussian we only give the correlation function (see Ref. [19]):

$$\langle g(0)g(k) \rangle = \sigma^2 e^{-2k\tau/\tau_0}, \tag{C13}$$

where τ_0 is the characteristic time for the RTS [the probability of having a pulse of length t is equal to $P(t) = (1/\tau_0)e^{-t/\tau_0}$].

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